Projects

for teaching mathematical physics through following its beginnings

Contents

Chapter 1

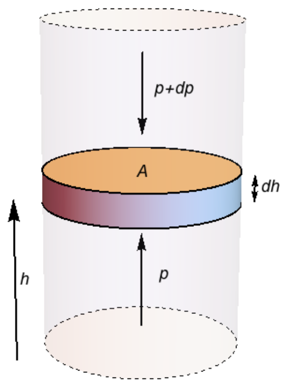
Project 1

1.1 A teen-friendly introduction to Euler

From weighing the air above the earth to the tricky and most simple methods made possible by using the most central number of all mathematics:

Assuming an isothermal atmosphere obeying Boyle's law Euler set out to find the change of atmospheric pressure with altitude and already in the way he set up the problem, he was led to almost all the properties of the exponential function that are known to constitute methods of mathematical physics; also he added the rest of its properties, the ones not related to earth's atmosphere.

Let's set up the equation for isothermal atmosphere at rest (to heights where g doesn't change appreciably)

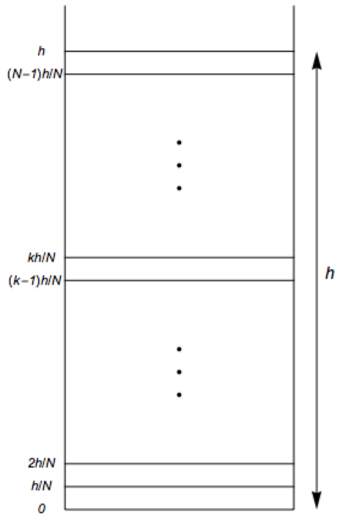


$$\begin{aligned} -(p+dp)A + p\,A &= dm\,g\,A \\ -dp\,A &= \rho\,g\,A\,dh. \end{aligned} \tag{1.1}$$

But $pV = nRT$ can be written as $p\frac{m}{\rho} = \frac{m}{m_{mol}}RT \Rightarrow p = \frac{RT}{m_{mol}}\rho$. Thus

$$dp = -\frac{m_{mol}\, g}{RT}\, p\, dh^1 \tag{1.2}$$

of course, now most of us would "separate variables" writing $\frac{dp}{p} = -\frac{m_{mol}\, g}{RT}\, dh$, we would integrate, we would use $\ln p$ and would end up with $p = p(0)\, e^{-\frac{m_{mol} g}{RT}\, h}$, with e being Euler's number. Let's see how Euler began.



(a) Our equation for dp gives (for h/N very small)

$$\begin{aligned} p(\frac{h}{N}) - p(0) &= -\frac{m_{mol}\, g}{RT}\,\frac{h}{N}\, p(0) \\ p(\frac{h}{N}) &= \left(1 - \frac{m_{mol}\, g}{RT}\,\frac{h}{N}\right) p(0). \end{aligned} \tag{1.3}$$

Similarly we have

$$\begin{aligned} p(\frac{2h}{N}) &= \left(1 - \frac{m_{mol}\, g}{RT}\,\frac{h}{N}\right) p(\frac{h}{N}) \\ &= \left(1 - \frac{m_{mol}\, g}{RT}\,\frac{h}{N}\right)^2 p(0) \end{aligned} \tag{1.4}$$

since $p(h) = p(N\,\frac{h}{N})$ we find that

[1] By introducing R, T, m_{mol} we are not using any more hindsight not available to Euler, that any 16 year old high school student can afford in our days

$$p(h) = \lim_{N\to\infty} \left(1 - \frac{m_{mol}\, g}{RT}\frac{h}{N}\right)^N p(0). \tag{1.5}$$

(b1) Setting $\frac{m_{mol}\, g}{RT} = C$ prove that $p(h)$ can be written as e^{-Ch}, with e defined as $\lim_{N\to\infty}(1+\frac{1}{N})^N$,(Keep record of any steps you consider as missing but plausible).

(b2) Evaluate $(1+\frac{1}{1})$, $(1+\frac{1}{2})^2$,..., $(1+\frac{1}{10})^{10}$. Does the formula $e = \lim_{N\to\infty}(1+\frac{1}{N})^N$ help to find e? How many digits are stabilized by the $N = 10$ step?

(b3) Expand $(1+\frac{1}{N})^N$ through Newton's binomial. Are there any missing steps if one just sets $N = \infty$ and gets $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots$? Assuming these steps will be filled in later or will be crosschecked through alternative calculations[2], answer if the above expressions lead to a trustable value for e if one starts like $1, 1+\frac{1}{1!}, 1+\frac{1}{1!}+\frac{1}{2!}, \ldots, 1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{10!}$. Can one prove that any promising behavior, like stabilization of digits, will continue to all higher orders?

Up to now we haven't even assumed that derivatives or integrals had ever been defined. In the way described, Euler could have solved the isothermal atmosphere problem even if Newton had not already invented calculus.

(c1) Suppose that Newton had already proved that $\frac{d}{dx}x^n = nx^{n-1}$ and suppose we want to solve the equation $dp = -Cpdh$ i.e. $\frac{dp}{dh} = -Cp$, by using as trial function a p of the form $p = a_0 + a_1 h + a_2 h^2 + \ldots$, (of course $a_0 = p(0)$). By matching powers of h on both sides prove again that $p(h) = p(0)\left(1 - \frac{Ch}{1!} + \frac{(-Ch)^2}{2!} + \ldots\right)$. Is this result a proof of $lim_{N\to\infty}(1-\frac{Ch}{N})^N = \left(1 - \frac{Ch}{1!} + \frac{(-Ch)^2}{2!} + \ldots\right)$?

Do you find some steps are missing? Is this result a doublechecking of the previous result $p(h) = p(0)e^{-Ch}$?

(c2) Prove that if a function $f(x)$ can be written in the form $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ then $a_n = \frac{1}{n!}\frac{d^n f(x)}{dx^n}\Big|_{x=0}$.

Are there any missing steps? Is the proof complete for at least the case where $f(x)$ is a polynomial of finite degree and not a polynomial of infinite degree like the power series we assumed?

[2] As Feynman puts it "if a calculation done by two or more wrong methods leads to the same result then the result is correct even if its proof is wrong"

In the above we didn't assume all functions are expandable in power series. We just assumed the solution to our equation was thus expandable (or not even that: we only assumed a solution was expandable. There may be others or at least we haven't proved there aren't). Actually, as we learn from Arnold [1], what Newton considered his methods to consist in was "something related to algebra in the way the theory of irrational numbers is related to the theory of rationals", i.e. he considered the way his methods were so powerful and universal was based on the fact that all equations, algebraic or differential or ..., accepted as solutions the newly found entities called "power series". They were introduced by him when after defining rational exponents through radicals he made a power series expansion of $(1+x)^{1/2}$ *by using for* $n = 1/2$ *the expansion* $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots+x^n$*, (as he himself said (see Arnold) he was ashamed to reveal to how many digits of accuracy he checked the validity of* $\sqrt{1+x} = 1+\frac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2+\ldots$*).*

OK, suppose Newton had introduced derivatives and one knew, at Euler's time, that $f(x+\varepsilon) \approx f(x)+\varepsilon Df(x)$ if ε is small.

(c3) Draw the figure of the column of height h divided in N equal slices but in a context unrelated to atmosphere etc.

$$\begin{aligned} f(\frac{h}{N}) &= f(0)+\frac{h}{N}f'(0) \\ &= (1+\frac{h}{N}D)f(0) \\ f(\frac{2h}{N}) &= (1+\frac{h}{N}D)f(\frac{h}{N}) = (1+\frac{h}{N}D)^2 f(0) \\ &\vdots \\ f(N\frac{h}{N}) &= ? \end{aligned} \tag{1.6}$$

where $Df(0)$ and $D^2f(0)$ are the first and second derivative respectively at $x=0$.Prove that

$$f(h) = \lim_{N\to\infty}(1+\frac{h}{N}D)^N f(0) \tag{1.7}$$

and assuming even now that D is not a number but an "operator", the following is still valid

$$\lim_{N\to\infty}(1+\frac{h}{N}D)^N f(0) = (1+\frac{h}{1!}D+\frac{h^2}{2!}D^2+\frac{h^3}{3!}D^3+\ldots)f(0) \tag{1.8}$$

prove that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots \quad (1.9)$$

What are the missing steps in "proving" as above that "all" functions can be expanded in power series?

(c4) Instead of trying to "prove" expandability of "all" functions let's only try alternative ways to prove it for some special functions based on special properties they may have. Thus like one can prove that $e^x = 1 + x + \frac{x^2}{2!} + \dots$ from $e^x = \lim_{N\to\infty}(1 + \frac{x}{N})^N$ one can prove $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$ from $\sin x = \sin N\frac{x}{N}$ and and $\sin\frac{x}{N} \approx \frac{x}{N}$[3] for N big, what about proving $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$ from $\cos x = \cos N\frac{x}{N}$ and $\cos\frac{x}{N} \sim 1$ for N big enough?

Can one prove $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \dots$ starting from $\ln(1+x) = N\ln(1+x)^{\frac{1}{N}}$ and taking $N \to \infty$? Can we crosscheck it with $\int_0^x \frac{dt}{1+t} = \int_0^x (1 - t + t^2 + \dots)\,dt$. What are the missing steps of each individually?

(d1) Applying De Moivre's theorem

$$(\cos\vartheta + i\sin\vartheta)^N = \cos N\vartheta + i\sin N\vartheta \quad (1.10)$$

to $\vartheta = \frac{x}{N}$ with $N \to \infty$ we get

$$\lim_{N\to\infty}(1 + i\frac{x}{N})^N = \cos x + i\sin x \quad (1.11)$$

Does this really mean that a meaning can assigned to e^{ix} and that it equals $\cos x + i\sin x$? Cross check it through power series for e^{ix}, $\cos x$, $\sin x$.

Here would be a good point to make a break for some homework problem e.g. related to the possibilities opened by $e^{ix} = \cos x + i\sin x$ to the solution of many differential equations often encountered in physics (e.g. in oscillations). Also to show homework problems related to solving $\frac{dp}{p} = -Cdh$ by treating $\int \frac{dp}{p}$ as an area, and e.g.proving the properties of ln from it (and also proving its base is $\lim_{n\to\infty}(1 + \frac{1}{n})^n$). But let's postpone these homework problems to go on with other things opened and pending by the development so far. Namely:

[3] Formulas for $\sin N\vartheta$, $\cos N\vartheta$ can be found from De Moivre's formula $(\cos\vartheta + i\sin\vartheta)^N = \cos N\vartheta + i\sin N\vartheta$ a formula resulting from $(\cos\vartheta + i\sin\vartheta)(\cos\vartheta' + i\sin\vartheta') = \cos(\vartheta + \vartheta') + i\sin(\vartheta + \vartheta')$ proved easily. Also show that $\sin\frac{x}{N} \sim \frac{x}{N}$ for N big results from identifying arcs with chords for small angles ϑ.

(a) Other operators with which "operator equations", like e.g. $\lim_{N\to\infty}(1+\frac{xD}{N})^N = 1 + xD + \frac{x^2}{2!}D^2 + \ldots$, may play a role.

(b) Other things that may be implied by taking power series as "infinite degree polynomials". Can, e.g. one set them equal to $c(x-r_1)(x-r_2)\ldots$ where r_1,r_2 ...are the roots of a function expandable in power series, e.g. $\sin x = 0$ has roots 0,$\pm\pi$,$\pm 2\pi$, ...etc. Is anything implied by setting $x - x^3/3! + x^5/5! + \ldots = cx(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\ldots$

Of course both of these issues were treated by Euler, as also was treated the issue of solving many differential equations (including all the linear differential equations with constant coefficients) through combinations of functions like $e^{(\alpha+i\beta)x}$, that we will leave for the homework. And also the issue[4] of finding as quickly as possible values of $\ln x$ for different values of x.

Let's start from the second issue (b);

Is it true that for a function $f(x)$ expandable in power series we have $f(x) = C(x-r_1)(x-r_2)(x-r_3)\ldots$ where r_1,r_2 ...are its roots? Well, C the coefficient of the highest degree doesn't even make sense but one can also try to write

$$f(x) = a_0(1-\frac{x}{r_1})(1-\frac{x}{r_2})\ldots \tag{1.12}$$

where a_0 is the coefficient of zero degree. Let's try all this for $\sin x$, or rather for $\frac{\sin x}{x}$ since $\sin x$ has zero a_0. Is the following relation true?

$$\frac{\sin x}{x} = (1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi})\ldots \tag{1.13}$$

Equivalently: is the following relation true?

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots = (1-\frac{x^2}{\pi^2})(1-\frac{x^2}{(2\pi)^2})(1-\frac{x^2}{(3\pi)^2})\ldots \tag{1.14}$$

(e1) Can we check it at special value of x?. At $x = \pi/2$ $\frac{\sin x}{x}$ becomes $\frac{2}{\pi}$. The infinite product formula becomes:

[4] Another pending/open issue is to collect the steps recorded as problematic or suspect or containing missing substeps. We will do this but let's postpone it until the appendix of page 26 to first get some more results trustable through just crosschecking etc.

$$\frac{2}{\pi} = (1-\frac{1}{2})(1+\frac{1}{2})(1-\frac{1}{4})(1+\frac{1}{4}) \tag{1.15}$$

So it is true that: $\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \ldots}$?

Is this an equation discovered? An equation already known and crosschecking part of what we want? A wrong equation disproving it? An equation checkable numerically?

Also let take the x^2 term of both sides of $1 - \frac{x^2}{3!} + \frac{x^4}{5!} \ldots = (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{(2\pi)^2}) \ldots$

We get: $\frac{1}{6} = \frac{\sum_{n=1}^{\infty} \frac{1}{n^2}}{\pi^2}$. Is it true that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$? Can we check this numerically? Can we crosscheck it with anything else? What does the comparison of x^4 term of the same equation gives?

Answer:The relations found are all true. And $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$, and the relation $\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \ldots}$ (also writable as $\sqrt{\pi} = \lim_{n \to \infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}$) had already been found before Euler (by Wallis), through methods not more difficult than relation like $\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx$ we all encountered as exercises when first learning about integration by parts, so it was really a crosscheck. The reader who can't wait to check these problems, can just google with something like *Wallis product proof.* The reader who can wait can remain here and continue to the paragraph below:

(e2) One can turn a product formula into a sum formula by taking logarithms of both sides. Also one can then remove logarithms by taking derivatives. By thus taking the "logarithmic derivative" of $\sin x = x(1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{2^2\pi^2}) \ldots$ prove that:

$$\frac{\cos x}{\sin x} = \frac{1}{x} + 2x\{\frac{1}{x^2 - \pi^2} + \frac{1}{x^2 - 2^2\pi^2} + \ldots\} \tag{1.16}$$

(frequently called "partial fraction expansion of $\cot x$ for obvious reasons. Of course it hardly constitutes a crosschecking of $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ if we rederive them from the above relation too, but do it. What are the loose steps now?

(e3) A probably better, or at least equivalent, crosschecking is to start from the last equation, the sum formula. When we make a partial fraction expansion like e.g. $\frac{3x+5}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$ we can find A, B as $\lim_{x \to 2}(x-$

2)$f(x)$, $\lim_{x\to 3}(x-3)f(x)$ when $f(x)=\frac{3x+5}{(x-2)(x-3)}$[5] Do we get the partial fraction expansion of $\cot x$ that we found if we start this way?

(e4) When comparing equal powers of x in the two sides of $\frac{\cos x}{\sin x}=\frac{1}{x}+2x\{\frac{1}{x^2-\pi^2}+\frac{1}{x^2-2^2\pi^2}+\ldots\}$ the right hand side is expanded in x-powers thus

$$\frac{1}{x^2-n^2\pi^2}=\frac{-1}{n^2\pi^2}\frac{1}{1-\frac{x^2}{n^2\pi^2}}=\frac{-1}{n^2\pi^2}\{1+\frac{x^2}{n^2\pi^2}+\frac{x^4}{n^4\pi^4}+\ldots\} \quad (1.17)$$

Can one avoid the very tedious approach of finding $f^{(n)}(x)$ with $f(x)=\frac{\cos x}{\sin x}-\frac{1}{x}$, $n=1,2,\ldots$ to find the coefficients of x^2, x^4,... needed?

One straightforward but less tedious would be to write $f(x)$ as $\frac{x\cos c-\sin x}{x\sin x}$, write it as quotient of two power series, and go brute force way, effecting the long division algorithm but applying from lower powers to higher (rather than from higher to lower as we usually do to derive polynomials and end up with a remainder instead of going on with an infinite series).

Try the above process in e.g. $\frac{1}{1-x}$ to see if it gives correctly $1+x+x^2+\ldots$ or in other simple case like e.g. $\frac{2x}{1-x^2}=(\frac{1}{1-x}-\frac{1}{1+x})$ also expandable by $\frac{1}{1-x}=1+x+x^2+\ldots$, $\frac{1}{1+x}=1-x+x^2+\ldots$.

Another way, a famous way actually, was found by Euler's contemporary Bernoulli:

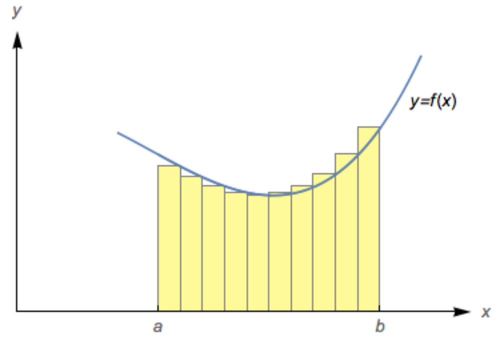
$$\begin{aligned}\frac{\cos x}{\sin x}&=\frac{(e^{ix}+e^{-ix})/2}{(e^{ix}-e^{-ix})/2i}=i(1+\frac{2e^{-ix}}{e^{ix}-e^{-ix}})\\&=i(1+\frac{2}{e^{2ix}-1})=i(1+\frac{1}{ix}\frac{2ix}{e^{2ix}-1})\end{aligned} \quad (1.18)$$

Unlike $\cot x$, $\frac{2ix}{e^{2ix}-1}$ is not problematic at $x=0$. If what has been tabulated (as "Bernoulli numbers" B_1,B_2, ...) is the expansion $\frac{x}{e^x-1}=B_0+\frac{B_1x}{1!}+\frac{B_2x}{2!}+\frac{B_3x}{3!}+\ldots$ then express $\sum_{n=2}^{\infty}\frac{1}{n^2}$, $\sum_{n=3}^{\infty}\frac{1}{n^4}$,$\sum_{n=4}^{\infty}\frac{1}{n^6}$,... in terms of Bernoulli numbers. Writing the equation for the expansion of $\frac{x}{e^x-1}$in the form $x=(e^x-1)(1+\frac{x}{1!}+\frac{x^2}{2!}B_2+\frac{x^3}{3!}B_3+\ldots)=(\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^2}{3!}+\ldots)(B_0+\frac{B_1x}{1!}+\frac{B_2x^2}{2!}+\ldots)$ prove that B's can be easily found

[5] The well known "rule of thumb"(quite literally so) of putting one's thumb over $x-2$ in $\frac{3x+5}{(x-2)(x-3)}$ and then setting $x=2$ to what remains visible to find A that is above $x-2$ in the RHS etc ...

through equations like $B_0 = 1$, $B_0 + 2B_1 = 0$, $B_0 + 3B_1 + 3B_2 = 0$, $B_0 + 4B_1 + 6B_2 + 4B_3 = 0 \ldots$, noticing that the coefficients we see are the binomial ones $\binom{n}{m}$. Using $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ prove this remark.

Before we go back to the issue of following up the manipulation of operator equations let's get back into their climate through the following warm up problem:



Considering the figure above, write the area as $\lim_{N\to\infty}\sum_{k=1}^{N} f(a+k\frac{b-a}{N})\frac{b-a}{N}$. Since $f(a+k\frac{b-a}{N}) = e^{k\frac{b-a}{N}D} f(a)$ it becomes:

$$\begin{aligned}
\text{Area} &= \lim_{N\to\infty}(1+e^{\frac{b-a}{N}D}+e^{2\frac{b-a}{N}D}+\ldots+e^{(N-1)\frac{b-a}{N}D})f(a)\frac{b-a}{N} \\
&= \lim_{N\to\infty}\frac{e^{N\frac{b-a}{N}D}-1}{e^{\frac{b-a}{N}D}-1}f(a)\frac{b-a}{N} \\
&= \lim_{N\to\infty}\frac{e^{(b-a)D}-1}{1+\frac{b-a}{N}D-1}f(a)(\frac{b-a}{N}) \\
&= \frac{e^{(b-a)D}-1}{\frac{b-a}{N}D}f(a)(\frac{b-a}{N}) = \frac{1}{D}(f(a+b-a)-f(a)) \\
&= \left[\frac{1}{D}f\right]_a^b = [D^{-1}f(x)]_{x=a}^{x=b} \qquad (1.19)
\end{aligned}$$

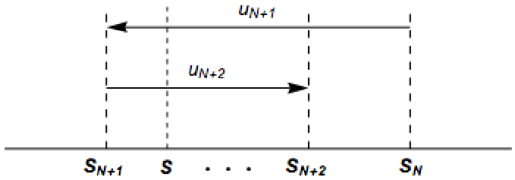
So areas can be found through antiderivatives; equivalently: the operation of integration is inverse differentiation.

Of course this fact, frequently called "the fundamental theorem of calculus", did not wait for such fancy ways before being discovered. As is often remarked in bibliography the theorem was very probably suggested by the mere fact that Galileo made diagrams of velocity versus time. In $v-t$ *diagrams area* $=\sum v\delta t = \sum \delta x =$ *displacement. But time derivative of displacement is velocity. Changing* $v(t)$ *to* $f(x)$, $\frac{dx}{dt}=v$ *becomes* $\frac{d(\text{area under } f(x) \text{ diagram})}{dx} = f(x)$.

Also a proof like the following was also very straightforward.

$$\begin{aligned}\int_a^b f'(x)\,dx &= \sum f'(x_k)(x_k - x_{k-1})\\ &= \sum \frac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}}(x_k - x_{k-1})\\ &= \sum (f(x_k) - f(x_{k-1}))\\ &= f(x_2) - f(x_1) + f(x_3) - f(x_2) + \ldots + f(x_N) - f(x_{N-1})\\ &= f(x_N) - f(x_1) = f(b) - f(a) = \left[\frac{1}{D}f'(x)\right]_a^b \qquad (1.20)\end{aligned}$$

Let's go back to Euler,



(f1) Suppose we want to sum the series $1-1/2+1/3-1/4+\ldots$ e.g. in order to check if it is really equal to $\ln 2$ as $\int_0^1 \frac{dx}{1+x} = \int_0^1 (1-x+x^2+\ldots)\,dx$ would suggest if it could be trusted[6]. Graphically the partial sums $1,\ 1-1/2,\ 1-1/2+1/3,\ \ldots$ as the figure above suggests, approach a number with errors smaller and smaller. The digits of the partial sums would do stabilize, first the tenths, then the hundredths, then the thousandths etc. (since after e.g. adding $-1/100$ the rest of the term up to infinity would not shift the place by more than $1/100$) but to stabilize the hundredths digit we would need at least 100 terms.So the series "converges slowly". For this, and for any other series like $a_0 - a_1 + a_2 - a_3 + \ldots$[7], Euler wrote, after defining the operators E by $Ea_n = a_{n+1}$ and Δ by $\Delta a_n = a_{n+1} - a_n$ and writing $\Delta = E - 1$, that $a_0 - a_1 + a_2 \ldots = (1 - E + E^2 + \ldots)a_0 = \frac{1}{1+E}a_0 = \frac{1}{2+\Delta}a_0 = \frac{1}{2(1+\frac{\Delta}{2})}a_0 = \frac{1}{2}(1 - \frac{\Delta}{2} + \frac{\Delta^2}{4}\ldots)a_0$. apply this to $1-1/2+1/3-1/4+\ldots$ and (to $1-1/3+1/5+\ldots$ for $\frac{\pi}{4}$, see footnote 7. Compare the speed of convergence

[6] a missing step besides the missing step $\int_0^1 (1-x+x^2+\ldots)\,dx = \int_0^1 dx - \int_0^1 x dx + \int_0^1 x^2 dx + \ldots$ is that the upper limit $x=1$ is a point where $\frac{1}{1+x} = 1 - x + x^2 + \ldots$ goes wrong, of course a single point does not contribute area to $\int_0^1 \frac{dx}{1+x}$ but is this enough of an excuse?

[7] if a_n is decreasing to zero the above remark about convergence apply as in $1-1/2+1/3\ldots$ e.g. $\int_0^1 \frac{dx}{1+x^2} = \int_0^1 (1-x^2+x^4+\ldots)\,dx$ give simply $\arctan x|_0^1 = 1 - 1/3 + 1/5 + \ldots$ i.e. $\frac{\pi}{4} = 1 - 1/3 + 1/5 + \ldots$

of the series you get to the speed of the series one finds with the more elementary trick of setting $\ln 2 = \ln \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = \ln(1+1/3) - \ln(1-1/3)$ and using $\ln(1+x) = x - x^2/2 + x^3/3 \ldots$. Notice also that Δ, $\Delta^2 \ldots$ etc. can be found from binomial coefficients (directly or through $\Delta^n = (E-1)^n$

(f2) What kind of results does the above trick give if applied to series like $a_0 - a_1 x + a_2 x^2 - \ldots$? Apply it e.g. to $1 - x + x^2 - x^3 + \ldots$, does it always accelerate the convergence ? Try e.g. $x = 1/2$, $x = 1/3$. Can we prove Euler's formula in the form $a_0 - a_1 + a_2 - a_3 + \ldots = \frac{1}{2}\{a_0 - \frac{a_1 - a_0}{2} + \frac{(a_2-a_1)-(a_1-a_0)}{4} \ldots\}$ without using operator methods? $a_0 - a_1 + a_2 - a_3 + \ldots$ can be written

$$\frac{1}{2}a_0 + \frac{1}{2}a_0 - \frac{1}{2}a_1 - \frac{1}{2}a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_2 - \frac{1}{2}a_3 \ldots$$
$$= \frac{1}{2}a_0 - \frac{1}{2}(a_1 - a_0) + \frac{1}{2}(a_2 - a_1) - \frac{1}{2}(a_3 - a_2) + \ldots$$

Now the series of differences let's again alternating signs. If we apply the same trick again we get the third term too of Euler's formula above.

(f3) For $1 - 1/2 + 1/3 - 1/4 + \ldots$ write the series as $\frac{1}{2}a_0 - \frac{1}{2}(a_1 - a_0) + \frac{1}{2}(a_2 - a_1) + \ldots$ and seeing that due to $a_n > a_{n+1}$ the new series is again alternating apply the argument of the previous page to see how many terms we need to stabilize the hundredths digit. Can we apply the same trick again to accelerate the convergence even more?

(f4) So Euler's trick seem to be based on breaking terms and rearranging the parts. What about the following argument regarding rearrangements of $1, -1/2, +1/3, -1/4, \ldots$.

Take any given number e.g. 105.6. Start adding all positive terms $1, 1/3, 1/5, \ldots$ until we pass over it and after we pass it start adding negative term $-1/2, -1/4$ until we pass below it. This is always possible because $1 + 1/2 + 1/3 + \ldots \to \infty$, $1 + 1/3 + 1/5 + \ldots \to \infty$, $1/2 + 1/4 + 1/5 + \ldots \to \infty$.(Proof: The first 9 terms of $1 + 1/2 + 1/3 + \ldots$ have sum $> \frac{1}{10} + \frac{1}{10} + \ldots + \frac{1}{10} = \frac{9}{10}$. The next 90 terms (the two digits ones) have sum $> \frac{1}{100} + \ldots + \frac{1}{100} = \frac{9}{10}$. The next 900 terms (the three digit ones) have sum $> \frac{900}{1000} = \frac{9}{10}$. so adding in such block we get sums $> \frac{9}{10} + \frac{9}{10} + \frac{9}{10} + \ldots$ which goes to infinity, (of course this step does not depend on taking groups 9, 90, 900,.... We have same number of digits in the group. we could have taken groups of 2, 4, 8 ... terms or of 3,9,27... etc. The same number of digits just made the sums more visible). But since the terms added when we make the passage from above to below or from below to above will be smaller and smaller as we go to higher and higher number of terms the difference from the target 105.6 will be getting less and less and will tend to zero since the terms added tend to zero. So by rearranging the terms we can make the series tend to any chosen target number.

Questions: Is this argument true? Does it also work with other series whose subseries of all positive terms and subseries of all negative terms, each separately,diverge? (i.e. can such series converge to any desired number by rearranging their terms?) If Euler's trick to accelerate convergence of series is based on breaking their terms in pieces and rearranging them, would such a theorem on the rearrangements of series affect the legitimacy of what we were saying about the Euler's trick? What is the effectiveness of Euler's trick to the series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots$?

(f5) Another closely related trick by Euler is the following:

$$x\sum_{n=0}^{\infty}(-1)^n a_n x^n = \sum_{n=0}^{\infty}(-1)^n a_n x^{n+1} = -\sum_{n=1}^{\infty}(-1)^n a_{n-1}x^n \tag{1.21}$$

Thus

$$\begin{aligned}(1+x)\sum_{n=0}^{\infty}(-1)^n a_n x^n &= a_0+\sum_{n=1}^{\infty}(-1)^n(a_n-a_{n-1})x^n = a_0+\sum_{n=1}^{\infty}(-1)^n\Delta a_{n-1}x^n\\ &= a_0-\sum_{n=0}^{\infty}(-1)^n(\Delta a_n)x^n \end{aligned}\tag{1.22}$$

Show that repeating the same procedure we go to

$$\sum_{n=0}^{\infty}(-1)^n a_n x^n = \frac{1}{1+x}[a_0-\frac{x}{1+x}\Delta a_0+(\frac{x}{1+x})^2\Delta^2 a_0+\ldots] \tag{1.23}$$

and that $1-x/2+x^2/3-\ldots$ becomes $\frac{1}{1+x}\{1+\frac{1}{2}(\frac{x}{1+x})+\frac{1}{3}(\frac{x}{1+x})^2+\ldots\}$. Are there any advantages in such a rewriting?

(f6) We have already said that $\Delta^n a_0$ can be found through binomial coefficient e.g. $\Delta^3 a_0 = a_3-3a_2+3a_1-a_0$ (proof: $\Delta^3 a_0=(E-1)^3a_0=(E^3-3E^2+3E-1)a_0=\ldots$). But writing the inverse relation $E^n=(1+\Delta)^n$ and applying Newton's binomial to $(1+\Delta)^n$ immediately has a much more interesting consequence:

$$a_n = E^n a_0 = (1+\Delta)^n a_0 = a_0 + n\Delta a_0 + \frac{n(n-1)}{2}\Delta^2 a_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 a_0 + \dots$$
$$= a_0 + n(a_1 - a_0) + \frac{n(n-1)}{2}(a_2 - 2a_1 + a_0)$$
$$+ \frac{n(n-1)(n-2)}{3!}(a_3 - 3a_2 + 3a_1 - a_0) + \dots \qquad (1.24)$$

Obviously the above relation proves that the graph of the polynomial of n^{th} degree

$$p(x) = a_0 + x(a_1 - a_0) + \frac{x(x-1)}{2}(a_2 - 2a_1 + a_0) + \dots$$
$$+ \frac{x(x-1)\dots(x-(n-1))}{n!}(a_n - na_{n-1} + \dots + (-1)^n a_0) \qquad (1.25)$$

passes from the $n+1$ points $(0, a_0)$, $(1, a_1)$,... (n, a_n), (an "interpolating polynomial" discovered by Newton, another immediate application of the Newton binomial formula $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \dots + b^n$ where a, b are operators).

The above has many interesting implications. One of them is: In a sense, finding a "summation formula" like e.g. $1 + 2 + \dots + n = \frac{n(n-1)}{2}$, $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ means equating an expression which like the left hand sides of these two summation formulas, can be defined for only integer (here positive integer) n with an expression which like their right hand sides can be defined for non-integer too.

We see that the formula $a_n = a_0 + n(a_1 - a_0) + \frac{n(n-1)}{2}(a_2 - 2a_1 + a_0) + \dots$ also has the same features. It can give a meaning to"a_x" for non-integer x by setting $a_x = a_0 + x(a_1 - a_0) + \frac{x(x-1)}{2}(a_2 - 2a_1 + a_0) + \dots$.

Two questions naturally arise:

(a) Can we use the above "Newton's interpolating formula" to re-obtain the usual summation formulas?

(b) Can we use this formula to give meaning to e.g. $1 + 1/2 + \dots + 1/n$ for non-integer n?

So:

(g1) By noticing Δa_0, $\Delta^2 a_0$, $\Delta^3 a_0$ ... vanish after a point for $a_n = n$,n^2, n^3, ... re-obtain the summation formula for $1+2+\ldots+n$, $1^2+2^2+\ldots+n^2$, $1^3+2^3+\ldots+n^3$. Before going to (g2) and see there a famous Euler treatment of the same issues let's review some good old brute force ways to solve the above problem: So brute force to (g1): To e.g. sum $1^3+2^3+\ldots+n^3$ try to write n^3 in the form $p(n+1)-p(n)$ where $p(n)$ is a polynomial in n. (If one can do that then $1^3+2^3+\ldots+n^3 = p(n+1)-p(n)+p(n)-p(n-1)+\ldots+p(2)-p(1) = p(n+1)-p(1)$ (the sum is called then "telescoping"). What degree N must the polynomial have if it exists? What are its coefficients? i.e. set $x^3 = a_N(x+1)^N + a_{N-1}(x+1)^{N-1}+\ldots+a_0$ and equate coefficients of equal powers of x.

Another remark on (g1): One can also notice that sum like $1\cdot 2+2\cdot 3+3\cdot 4+\ldots+(n-1)\cdot n$ or $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\ldots+(n-2)\cdot(n-1)\cdot n$ etc are easier to do because their terms e.g. $(x-2)(x-1)x$ are already in the form $p_4(x+1)-p_4(x)$ e.g. the $p_4(x)$ needed in for $(x-2)(x-1)x$ is, as easily checked , as $\frac{1}{4}x(x-1)(x-2)(x-3)$.

So one can try to e.g. x^3 as a combination of such polynomials $(p_4, p_3, p_2, p\ldots)$. Show also that analogous things hold for e.g. $\frac{1}{n(n-1)(n-2)}$ so one can telescope and sum things like $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots\frac{1}{(n-2)(n-1)n}$. Can these help to at least find an approximate numerical value for $\sum_{n=1}^{\infty}\frac{1}{n^2}$? What remains if we subtract $\frac{1}{n(n-1)}$ from $\frac{1}{n^2}$ goes like $\frac{1}{n^3}$ as $n\to\infty$. Does this help in estimating numerically $\sum\frac{1}{n^2}$ by e.g. bounding the remainder by an integral?

(g2) What about using Newton's interpolation formula to sum $1+\frac{1}{2}+\ldots+\frac{1}{n}$? Now Δa_0, $\Delta^2 a_0$, $\Delta^3 a_0$ ... do not terminate. Do they have pattern? What about $1+\frac{1}{2^k}+\ldots+\frac{1}{n^k}$? What about $1\cdot 2\ldots\cdot n$?

Now let's go to some famous ways Euler treated these subjects (the first one was also found by McLaurin).

(h1) We have seen that $e^{aD}f(x) = f(x+a)$. Then $\Delta f(x) = f(x+1)-f(x) = e^D f(x)-f(x) = (e^D-1)f(x)$. Thus $\Delta = e^D-1$.

Like integration is inverse to differentiation, summation is inverse to taking differences. Now $\Delta^{-1} = \frac{1}{e^D-1} = \frac{1}{D}\frac{D}{e^D-1}$.

But $\frac{x}{e^x-1}$ has already tabulated coefficients for its power series the Bernoulli numbers B_0,B_1, B_2,. . . , so

$\frac{D}{e^D-1} = B_0+\frac{1}{1!}B_1 D+\frac{1}{2!}B_2 D^2+\ldots$ (B_0 is easily seen to be 1).

Thus $\Delta^{-1} = \frac{1}{D}+\frac{1}{1!}B_1+\frac{1}{2!}B_2 D+\frac{1}{3!}B_3 D^2+\ldots$

Use this to prove the celebrated Euler-McLaurin formula [2].

$$f(0)+f(1)+\ldots+f(n)=\int_0^n f(x)\,dx+\frac{1}{2}[f(n)+f(0)]$$
$$+\frac{1}{12}\{f'(n)-f'(0)\}-\frac{1}{720}\{f'''(n)-f'''(0)\}+\ldots \tag{1.26}$$

Then prove the summation formula for$1+2+\ldots+n$, $1^2+2^2+\ldots+n^2$, $1^3+2^3+\ldots+n^3$ through it.

Let's continue with Euler's treatment of $1+\frac{1}{2}+\ldots+\frac{1}{n}$:

(h2) Let's start from $\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}$ instead since we will need a step that involves the infinite sum of terms and $\sum_{n=1}^\infty \frac{1}{n}=\infty$ whereas $\sum_{n=1}^\infty \frac{1}{n^2}<+\infty$[8] Also, equally easily proved, $\sum \frac{1}{(n+x)^2}<+\infty$ for all positive reals x.

Now, notice that $\sum_{n=1}^\infty[\frac{1}{n^2}-\frac{1}{(n+x)^2}]$ *is a summation formula for* $\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{N^2}$ *in the sense that it is defined for non integer x too, and that when x is a positive integer, N it reduces to* $\frac{1}{1^2}+\ldots+\frac{1}{N^2}$. *Proof: e.g for* $N=3$, $\sum_{n=1}^\infty[\frac{1}{n^2}-\frac{1}{(n+3)^2}]=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots-\frac{1}{4^2}-\frac{1}{5^2}-\ldots=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}$.

Q.E.D! That's all.[9]

(h3) From $f_1(x)=\sum_{n=1}^\infty\left(\frac{1}{n}-\frac{1}{n+x}\right)$

and writing $\frac{1}{n+x}$ as $\frac{1}{n(1+\frac{x}{n})}=\frac{1}{n}(1-\frac{x}{n}+\frac{x^2}{n^2}-\frac{x^3}{n^3}+\ldots)$ and keeping track of possibly suspect steps prove that $f_1(x)=x\sum_{n=1}^\infty\frac{1}{n^2}-x^2\sum_{n=1}^\infty\frac{1}{n^3}+\ldots$.

[8] Checkable in many ways: e.g. $\sum\frac{1}{n^2}<\int_2^\infty\frac{dx}{x^2}=\frac{1}{2}$. Also $\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{9^2}<9\cdot\frac{1}{1^2}$, $\frac{1}{10^2}+\ldots+\frac{1}{99^2}<\frac{90}{10^2}$, $\frac{1}{100^2}+\ldots+\frac{1}{999^2}<\frac{900}{100^2}\ldots$thus $\sum_{n+1}^\infty\frac{1}{n^2}<9(\frac{1}{1}+\frac{1}{10}+\frac{1}{100}+\ldots)=9\cdot\frac{1}{1-1/10}=10$

[9] Combining with $\sum_{n=1}^\infty\frac{1}{n^2}=\frac{\pi^2}{6}$ we see that $\frac{\pi^2}{6}-\sum_{n=1}^\infty\frac{1}{(n+N)^2}$ is the summation formula for $\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{N^2}$. Does this most simple trick work for $1+\frac{1}{2}+\ldots+\frac{1}{N}$? Well $\sum_{n=1}^\infty\frac{1}{n}$, $\sum_{n=1}^\infty\frac{1}{n+x}$ are both $+\infty$ but $\sum_{n=1}^\infty(\frac{1}{n}-\frac{1}{n+x})$ is not divergent, it is $\sum_{n=1}^\infty\frac{n+x-n}{n(n+x)}=x\sum_{n=1}^\infty\frac{1}{n(n+x)}$ and $\frac{1}{n(n+x)}$ goes like $\frac{1}{n^2}$ as $n\to\infty$ so it is $<+\infty$. So $x\sum_{n=1}^\infty\frac{1}{n(n+x)}$ is a summation formula for $1+\frac{1}{2}+\ldots+\frac{1}{N}$.And like one can find e.g. $\int_A^B\frac{dx}{(x+A)(x+B)}$ (through partial fractions) in term of ln's are confined $\sum_n^N\frac{1}{(n+A)(n+B)}$ (through partial fractions in terms of $a\sum\frac{1}{n(n+a)}$, $b\sum\frac{1}{n(n+b)}$'s) the discrete analogous of ln's. For all such results see [2].

Thus $f_1'(0) = \zeta(2)$, $f_1''(0) = -2!\zeta(3)$, ... where the "zeta-function" $\zeta(s)$ is defined as $\sum_{n=1}^{\infty} \frac{1}{n^s}$.

Also prove that $f_1'(x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2}$,$f_2''(x) = -2! \sum_{n=1}^{\infty} \frac{1}{(n+x)^3}$

Also prove that: $f_1(x+1) = f_1(x) + \frac{1}{x+1}$, $-f_1'(x+1) = -f_1'(x) + \frac{1}{(x+1)^2}$, ... Also prove that $f_2(x+1) = f_2(x) + \frac{1}{(x+1)^2}$, ...

(h4) Also prove that: $\int_1^{x+1} f_1(t)\,dt - \int_0^x f_1(t)\,dt = \ln(x+1)$,

thus prove $e^{\int_1^{x+1} f_1(t)\,dt} = (x+1)e^{\int_0^x f_1(t)\,dt}$.

Thus $f_1(x)$ is the logarithmic derivative (within an additive constant) of an appropriately defined function $G(x)$ fulfilling $G(x+1) = (x+1)G(x)$ and $G(0) = 1$.[10] Thus of a function which at $x = n$, an integer, becomes $n!$. Thus a solution to the problem of a summation formula for $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ also creates a solution to the problem of "multiplication formula" for $n!$ and vice versa, had one known a solution to $G(x+1) = (x+1)G(x)$, $G(0) = 1$, (which would of course fulfill $G(n) = n!$) one get to $\ln G(x+1) = \ln(x+1) + \ln G(x)$ thus $\frac{G'(x+1)}{G(x+1)} = \frac{1}{x+1} + \frac{G'(x)}{G(x)}$ and thus $\frac{G'(n)}{G(n)}$ would equal $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ plus a constant (not necessarily equal to 0) since fixing the difference $\frac{G'(x+1)}{G(x+1)} - \frac{G'(x)}{G(x)}$ equals $\frac{1}{x+1}$ allows an arbitrary constant for $\frac{G'(x)}{G(x)}$.

(h5) A much more familiar function fulfilling $g(x+1) = (x+1)g(x)$ we all have seen ever since our first exercise in integration by parts (in e.g. recursive formulas like $\int e^{-t}t^{n+1}\,dt = -t^{n+1}e^{-t} + (n+1)\int e^{-t}t^n\,dt$

thus gives $\int_0^{\infty} e^{-t}t^{n+1}\,dt = [-t^{n+1}e^{-t}]_0^{\infty} + (n+1)\int_0^{\infty} e^{-t}t^n\,dt$.

For $n > -1$ the term $[-t^{n+1}e^{-t}]_0^{\infty}$ is 0,

$\int_0^{\infty} e^{-t}t^n\,dt$ for $n = 0$ becomes 1

and $\int_1^{\infty} e^{-t}t^n\,dt = n!$ for n integer,

while $\int_1^{\infty} e^{-t}t^{x+1}\,dt = (x+1)\int_0^{\infty} e^{-t}t^x\,dt$ works for all $x > -1$.

So the question arises of whether this function and the function $G(x)$ defined on the previous page are the same. (The function defined is of course the so well known Gamma function or rather, not quite but with a $x \to x+1$ redefinition, $\Gamma(x)$ is defined as $\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}\,dt$, thus $\Gamma(x+1) = x\Gamma(x)$ NOT $(x+1)\Gamma(x)$. Of course this function too may defined and studied by Euler).

[10] Check that such an appropriate definition is $G(x) = e^{\int_0^x (f_1(t)-c)\,dt}$ with $c = \int_0^1 f_1(t)\,dt$.

(h6) Two pages ago we saw an utterly simple way to make a summation formula for $1+\frac{1}{2}+\ldots+\frac{1}{n}$ by using sum extending to infinity. Then we went and made a product formula of $n!$ by taking exponentials. The question arises if we could directly make a multiplicative analog of the simple trick; if yes the question also arises whether the function thus defined and our previous solution are the same (or actually the previous two solutions, we also met. The Gamma function was defined through an integral rather than through that trick). Let's see:

Can we define x as $\frac{1\cdot 2\cdot 3\cdot\ldots\infty}{(x+1)(x+2)\ldots\infty}$? Setting x equal to an integer e.g. $x=5$ we cancel everything at the bottom with everything at the top except $1\cdot 2\cdot 3\cdot 4\cdot 5$ so it works but unlike in the additive case $\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+x})$ where $\sum_{n=1}^{\infty}\frac{1}{n}$, $\sum_{n=1}^{\infty}\frac{1}{n+x}$ were ∞ but their difference was not, now there is something fishy: Making e.g. the fraction into a difference by taking ln's we get

$$\sum_{n=1}^{\infty}\{\ln n-\ln(n+x)\}=-\sum_{n=1}^{\infty}\ln(1+\frac{x}{n})=-\sum_{n=1}^{\infty}\{\frac{x}{n}-\frac{1}{2}\left(\frac{x}{n}\right)^2+\frac{1}{3}\left(\frac{x}{n}\right)^3+\ldots\}$$

and the first term still makes the sum diverge. It is NOT like $\sum(\frac{1}{n}-\frac{1}{n+x})=\sum\frac{x}{n(n+x)}$ that was $<+\infty$.

Here's how Euler made for $x!$ a fraction of convergent entities:

First take x a positive integer: (and n <u>any</u> fixed positive integer)

$$\begin{aligned}x! &= \frac{1\cdot 2\ldots x(x+1)\ldots(x+n)}{(x+1)\ldots\ldots(x+n)}=\frac{1\cdot 2\ldots n}{(x+1)\ldots(x+n)}(n+1)\ldots(n+x)\\ &= \frac{1\cdot 2\ldots n}{(x+1)\ldots(x+n)}n^x\left[\frac{n+1}{n}\cdot\frac{n+2}{n}\cdot\ldots\cdot\frac{n+x}{n}\right]\\ &= \frac{n!\,n^x}{(x+1)\ldots(x+n)}\left(1+\frac{1}{n}\right)\ldots\left(1+\frac{x}{n}\right) \qquad (1.27)\end{aligned}$$

still keeping x an integer and taking $n\to\infty$ we get:

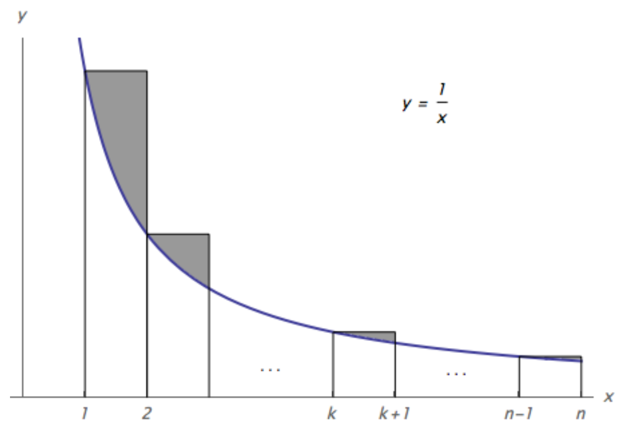
$$x!=\lim_{n\to\infty}\frac{n!\,n^x}{(x+1)\ldots(x+n)}$$

We note that with this definition as starting point it's easy to prove that for e.g. $x=3$ we get $x!=1\cdot 2\cdot 3$.

We also note that it is still easy to prove that $(x+1)! = (x+1)x!$ for non-integer x too. But let's take its ln

$$\begin{aligned}\ln x! &= \lim_{n\to\infty}\left[x\ln n + \ln\left(\frac{1}{x+1}\cdot\frac{2}{x+2}\cdots\frac{n}{x+n}\right)\right]\\ &= \lim_{n\to\infty}\left[x\ln n - \ln(1+x) - \ln(1+\frac{x}{2}) - \ldots - \ln(1+\frac{x}{n})\right]\\ &= \lim_{n\to\infty} x\left[\ln n - \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\right] + \frac{1}{2}\sum_{n=1}^{\infty}x^2\frac{1}{n^2} - \frac{1}{3}\sum_{n=1}^{\infty}x^3\frac{1}{n^3} + \ldots\end{aligned} \tag{1.28}$$

But $\ln n - \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)$ is convergent and setting $S_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}$, we have $\lim_{n\to\infty}(S_n - \ln n) = \gamma$. This limit γ is called the Euler's constant. Before continuing, we try to justify its existence using the following diagram:



(i) Using rectangles in the above diagram, and starting from the obvious inequality $\frac{1}{k+1} < \int_k^{k+1}\frac{1}{x}\,dx < \frac{1}{k}$, we deduce that $\frac{1}{n} + \ln(n) \le S_n \le 1 + \ln(n)$ (and hence it follows that $\lim_{n\to\infty}\frac{S_n}{\ln(n)} = 1$, so we could say, informally, that S_n grows logarithmically).

(ii) If $A(k)$ is the area of the shaded region contained in the rectangle between k and $k+1$, we can see that

$$\frac{1}{2}(1-\frac{1}{n}) < \sum_{k=1}^{n-1}A(k) < 1-\frac{1}{n}$$

and hence, computing $A(k)$, it can be shown that

$$\frac{1}{2}(1+\frac{1}{n}) < S_n - \ln(n) < 1$$

which proves that as $n \to \infty$ the sequence $t_n = S_n - \ln(n)$ remains bounded. Clearly, $0.5 < t_n < 1$.

(iii) Finally, it can easily be seen from the figure above that $t_{n+1} - t_n = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x}\, dx < 0$, hence the sequence t_n is decreasing, and as we have seen before it is also bounded from below. Does that prove it is convergent? Answer: Yes, it does! The properties of t_ns that we have proved so far above do show that if we break the interval $(0, 1)$ in ten equal subintervals, then there will be a lowest subinterval still containing t_ns and below which we don't see anymore of them. Say it's the sixth above $(0, 0.1)$, [11] i.e. $(0.5, 0.6)$. Break this last interval in ten equal subintervals and take again the lowest one still containing t_ns and below which we don't see any more of them. Say it's the 8^{th} above $(0.50, 051)$, i.e. $(0.57, 058)$. Then, again, break this interval in ten equal subintervals ... and so forth and so on; finally take the number formed by the new tenths' digit, hundredths' digit etc etc added, just like we formed 0.57 above. The number formed is a number to which the bounded sequence we had started with, converges because being decreasing it cannot get back away from the intervals reached by terms of it, and also it approaches it to distances first less than tenths, then less than hundredths, then less than thousandths ... etc etc. [12]

So this definition of $\ln x!$ is OK. Taking now the derivative of this log we find it equals: $-\gamma + x \sum_{n=1}^{\infty} \frac{1}{n^2} - x^2 \sum_{n=1}^{\infty} \frac{1}{n^3} + \ldots$, so it does agree with the definition that ended up like $x \sum \frac{1}{n^2} - x^2 \sum \frac{1}{n^3} + \ldots$ plus a possible constant

(h7) OK, up to now, we have defined in parts (h3) and (h4) the function$G(x) = x!$, such that its logarithmic derivative is given by $f_1(x) - c = \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+x} - c$, with the above power series expansion, as it can now easily be shown that $c = \int_0^1 f_1(x)\, dx = \gamma$.

In part (h6), $x!$ was defined as the limit of an infinite product, due to Euler, and it was seen that the power series expansion of its logarithmic derivative was identical to the one obtained for the previous case.

In part (h5), following a third approach we had $x! = \Gamma(x+1) = \int_0^{\infty} e^{-t} t^x\, dx$. As it is not easy to check directly if the logarithmic derivative of $\Gamma(x+1)$ has an identical form with the previous cases, we check at least, for consistency, the constant term, i.e. if $\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma$. Since $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dt$ and $\Gamma(1) = 1$, can we prove that $\frac{d}{dx} \int_0^{\infty} e^{-t} t^{x-1}\, dt|_{x=1} = -\gamma$? or,equivalently that $\int_0^{\infty} e^{-t} \ln t\, dt = -\gamma$?

[11] By the inequalities we have seen, can we prove it's really the sixth above $(0, 0.1)$?

[12] Is the above argument a proof for the specific decreasing sequence we have? For all decreasing sequences bounded from below? Are steps missing? See also (or see later in case the sequel of what we were reading feels more attractive than the proof or "proof" we have seen) the Appendix on rigor at the end of this chapter at issue d.

Let's see: Can we prove $\int_0^\infty e^{-t}\ln t\,dt = \lim_{n\to\infty}\left(\ln n - \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\right)$? The left hand side integral is undoable as an indefinite integral but $\int t^n \ln t\,dt$ is doable for integer n's. So let's try our luck by writing the left hand side as $\lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n \ln t\,dt$.

Transform the complicated term $\left(1-\frac{t}{n}\right)^n t\ln t$ by setting $1-\frac{t}{n}=\omega$. Then the expression becomes

$$\begin{aligned} n\int_0^1 \omega^n \ln n(1-\omega)\,d\omega &= n\ln n\int_0^1 \omega^n\,d\omega + n\int_0^1 \omega^n \ln(1-\omega)\,d\omega \\ &= \frac{n}{n+1}\ln n - n\int_0^1 \omega^n\left(\omega+\frac{\omega^2}{2}+\frac{\omega^3}{3}+\ldots\right)d\omega \\ &= \frac{n}{n+1}\ln n - \frac{n}{n+2} - \frac{n}{n+3}\frac{1}{2} - \frac{n}{n+4}\frac{1}{3} - \ldots \end{aligned} \tag{1.29}$$

As $n\to\infty$thus where goes $\ln n - 1 - \frac{1}{2} - \ldots$. Keep record of missing steps. That was a consistency check but we still haven't proved that the definition of $x!$ through an integral and through an infinite product agree! They might differ by a factor who is a periodic function of period 1.

(h8) Can we use the integral definition to go to the $\lim_{n\to\infty}\frac{n!n^x}{(x+1)(x+2)\ldots(x+n)}$ definition? e.g. by treating e^{-t} in the integral just like we treated it above in the integral $\int_0^\infty e^{-t}\ln t\,dt$.

Let's see: Write $\int_0^\infty e^{-t}t^x\,dt$ as $\lim_{n\to\infty}\int_0^n\left(1-\frac{t}{n}\right)^n t^x\,dt$. Is this equal to $\lim_{n\to\infty}\frac{n!\,n^x}{(x+1)\ldots(x+n)}$?

Set $1-\frac{t}{n}=\omega$ then the integral becomes: $n^{x+1}\int_0^1\omega^n(1-\omega)^x\,d\omega$.

To transfer the difficulty to the factor that has integer exponent and is tractable set $\omega=1-z$. Then the integral becomes $n^{x+1}\int_0^1(1-z)^n z^x\,dz$. An integration by parts makes this as

$$\frac{n^{x+1}}{x+1}(1-z)^n z^{x+1}\Big|_0^1 + \frac{n^{x+1}n}{(x+1)}\int_0^1 (1-z)^{n-1}z^{x+1}\,dz.$$

For $x>-1$ the first term is 0. Repeating a similar integration by parts the integral becomes:

$$\frac{n^{x+1}n(n-1)}{(x+1)(x+2)}\int_0^1(1-z)^{n-2}z^{x+2}\,dz.$$

Finally we end up with

$$\frac{n^{x+1}n!}{(x+1)\dots(x+n)}\int_0^1 z^{x+n}\,dz = \frac{n^x n!}{(x+1)\dots(x+n)}\frac{n}{n+x+1}$$

which apart from $\frac{n}{n+x+1}$ that tends to 1 as $n\to\infty$ which is what we wanted.

Having proved that all the definitions we've seen really refer to the same object let's see if each of them, maybe reveals some properties not obvious from another of them.

(h9) The definition $\Gamma(x) = \lim_{n\to\infty}\frac{n!n^{x-1}}{x(x+1)\dots(x+n-1)}$ shows that $\frac{1}{\Gamma(x)}$ has roots at $x = 0$, $x = -1$, $x = -2$ .... So it misses the roots +1,+2,... to have all the roots of $\sin\pi x$. But if we multiply it with $\frac{1}{\Gamma(1-x)}$ then it does acquire them. So maybe $\frac{1}{\Gamma(x)\Gamma(1-x)}$ and $\sin\pi x$ differ by a constant factor (or maybe by something with no zeros if such a thing exists).

Looking more closely:

$$\frac{1}{\Gamma(x)\Gamma(1-x)} = \lim_{n\to\infty}\frac{n!n^{x-1}n^{-x}n!}{x(x+1)\dots(x+n-1)(1-x)(2-x)\dots(n-x)}$$
$$\lim_{n\to\infty}\frac{1}{x(1-\frac{x^2}{1^2})(1-\frac{x^2}{2^2})\dots(1-\frac{x^2}{(n-1)^2})} = \frac{\sin\pi x}{\pi} \tag{1.30}$$

to the infinite product for $\sin\pi x$ we saw Euler had written.

Can this be proved directly from the integral definition? Is it true that:

$$\int_0^\infty e^{-t}t^{x-1}\,dt\int_0^\infty e^{-t}t^{-x}\,dt = \frac{\pi}{\sin\pi x}? \tag{1.31}$$

Before trying the general case let's try the special case $x = 1/2$ where $\Gamma(x)$ and $\Gamma(1-x)$ are equal. can one prove the following relation suggested by the product formula: $[\Gamma(1/2)]^2 = \pi$?, i.e. can we prove directly that $\int_0^\infty e^{-t}t^{-1/2}\,dt = \sqrt{\pi}$?

Setting $t = x^2$ then becomes $2\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}$ i.e. $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$. following Euler, do this by writing it as $\int_{-\infty}^{+\infty} e^{-x^2}\,dx\int_0^\infty e^{-y^2}\,dy = \pi$ and rewriting thus as $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy$ and transforming to polar variables...

To go to the general case of $\Gamma(x)\Gamma(1-x)$ let's study, through a similar trick involving polar variables, the product $\Gamma(a)\Gamma(b)$. Prove that

$\Gamma(a)\Gamma(b) = \Gamma(a+b)2\int_0^{\pi/2}\cos^{2a-1}\theta\sin^{2b-1}\theta\, d\theta$. Writing $u = \cos^2\theta$ also prove $\int_0^{\pi/2}\cos^{2a-1}\theta\sin^{2b-1}\theta\, d\theta = \int_0^1 u^{a-1}(1-u)^{b-1}\, du$.

So $\Gamma(x)\Gamma(1-x) = \int_0^1 u^{x-1}(1-u)^{-x}\, du$. However this integral does not seem easily reducible to $\frac{\pi}{\sin \pi x}$. Yet the formula suggests integrals like e.g.$\int_0^1 \frac{dx}{x^{2/3}\sqrt[3]{1-x}}$ are doable. But through what method? Only through product formulas for related Gamma functions?

Another manipulation worth trying in the product formula is multiplying $\Gamma(x)$ i.e. $\Gamma(2x/2)$ with $\Gamma(x+\frac{1}{2})$ i.e. $\Gamma(\frac{2x+1}{2})$ to get something involving $\Gamma(2x)$ since the factors $(2x+1)(2x+2)\ldots(2x+n)$, some of 2s are factored out, can be separated into $(x+1)(x+2)\ldots$ and $(x+\frac{1}{2})(x+\frac{3}{2})\ldots$. Show one ends up with:

$$2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2}) = \sqrt{\pi}\Gamma(2x)$$

Can be proved through the definition of $\Gamma(x)$ as an integral?.

Show similarly, that the following holds:

$$\Gamma(x)\Gamma(x+\frac{1}{3})\Gamma(x+\frac{2}{3}) = (2\pi)3^{1/2-3x}\Gamma(3x)$$

in general

$$\Gamma(x)\Gamma(x+\frac{1}{n})\Gamma(x+\frac{2}{n})\ldots\Gamma(x+\frac{n-1}{n}) = (2\pi)^{\frac{n-1}{2}}n^{1/2-nx}\Gamma(nx)$$

(h10) We saw that $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$. Easily from this find $\int_0^\infty e^{-\alpha x^2}\, dx$ as a function of α. And if we differentiate what we found with respect to α we find $\frac{d}{d\alpha}\int_0^\infty e^{-\alpha x^2}\, dx$ i.e. we find $-\int_0^\infty x^2 e^{-\alpha x^2}\, dx$ and similarly we can find $\int_0^\infty x^4 e^{-\alpha x^2}\, dx$ by differentiating again.

Also $\int_0^\infty e^{-\alpha x^2} x\, dx$ is easily found (with no tricks) $\int e^{-\alpha x^2} x\, dx$ is doable even as indefinite integral and similarly we find $\int_0^\infty x^3 e^{-\alpha x^2}\, dx$, $\int_0^\infty x^5 e^{-\alpha x^2}\, dx$, ...

By completing squares to remove the linear term find also $\int_0^\infty e^{-\alpha x^2+\beta x}\, dx$, $\alpha > 0$.

One way Laplace found to study $x!$, or rather its $\ln$, for high values of x (integer or not) was the following:

$$x! = \int_0^\infty e^{-t} t^x \, dt = \int_0^\infty e^{-t + x\ln t} \, dt$$

Write the exponent as an expansion in powers of the distance from its minima so as not to have a linear term. That minimum is at the solution, for t, of the equation: $\frac{d}{dt}(-t + x\ln t) = 0 \to -1 + \frac{x}{t} = 0 \to t = x$. Shift t to u by setting $t = x + u$, then $-t + x\ln t$ becomes

$$
\begin{aligned}
&- x - u + x\ln(x+u) = -x - u + x\ln[x(1+\frac{u}{x})] \\
&- x - u + x\ln x + x\ln(1+\frac{u}{x}) = -x - u + x\ln x + x\frac{u}{x} - \frac{x}{2}\frac{u^2}{x^2} + x\frac{1}{3}\frac{u^3}{x^3}\dots \\
&= -x + x\ln x - \frac{1}{2x}u^2 + \frac{1}{3x^2}u^3 - \frac{u^4}{4x^3} + \dots \qquad (1.32)
\end{aligned}
$$

Thus

$$\int_0^\infty e^{-t + x\ln t} \, dt = e^{-x + x\ln x} \int_{-x}^\infty e^{-\frac{u^2}{2x} + \frac{u^3}{3x^2} + \dots} \, du$$

thus

$$\ln x! = -x + x\ln x + \ln \int_{-x}^\infty e^{-\frac{u^2}{2x}} \, e^{(\frac{u^3}{3x^2} - \frac{u^4}{4x^3})} \, du$$

Does this help at all in anything? $e^{-u^2/2x}$ is a very broad exponential but if we rescale u setting equal to $x\nu$ the integral becomes:

$$x \int_{-1}^\infty e^{-\frac{1}{2}x\nu^2} \, e^{x(\frac{\nu^3}{3} - \frac{\nu^4}{4} + \dots)} \, d\nu.$$

When x is big $e^{\frac{-1}{2x\nu^2}}$ is narrowly peaked around $\nu = 0$ and hopefully $\int_{-1}^\infty (\dots)\, d\nu$ can be just set equal to $\int_{-\infty}^{+\infty} (\dots)\, d\nu$ with error that is exponentially small, e.g.

$$\int_{-\infty}^{-1} e^{-x\nu^2/2} \, d\nu \le \int_{-\infty}^{-1} e^{-x|\nu|/2} \, d\nu = \frac{2}{x} e^{-x}$$

Let's see if we get terms that are not so small: The integral is

$$x \int_{-\infty}^{+\infty} e^{\frac{-1}{2}x\nu^2} \left(1 + x\left(\frac{1}{3}\nu^3 - \frac{1}{4}\nu^4 + \dots\right) + \frac{x^2}{2!}\left(\frac{1}{3}\nu^3 - \frac{1}{4}\nu^4 + \dots\right)^2 + \dots \right) d\nu$$

All terms are of the doable form $\int_{-\infty}^{+\infty} \nu^n e^{-x\nu^2/2} \, d\nu$. The leading term in powers of x are:

$$x \int_{-\infty}^{+\infty} e^{\frac{-1}{2}x\nu^2}\, d\nu - x \cdot x \int_{-\infty}^{+\infty} \frac{\nu^4}{4} e^{\frac{-1}{2}x\nu^2}\, d\nu + x \cdot x^2 \int_{-\infty}^{+\infty} \frac{\nu^6}{9} e^{\frac{-1}{2}x\nu^2}\, d\nu$$
$$= x\sqrt{\frac{2\pi}{x}} - \frac{1}{4}x^2 \frac{3\sqrt{2\pi}}{x^2\sqrt{x}} + x^3 \frac{15\sqrt{2\pi}}{18x^3\sqrt{x}} = \sqrt{2\pi x}\left\{1 + \frac{1}{12x}\right\} \qquad (1.33)$$

Thus

$$\ln x! = -x + x \ln x + \ln\left[\sqrt{2\pi x}\left\{1 + \frac{1}{12x} + \ldots\right\}\right]$$
$$= \ln\sqrt{2\pi} - x + x\ln x + \frac{\ln x}{2} + \ln\left\{1 + \frac{1}{12x} + \ldots\right\} \qquad (1.34)$$

Expanded and rearranged into descending order this is:

$$x \ln x - x + \frac{\ln x}{2} + \ln\sqrt{2\pi} + \frac{1}{12x} + \ldots$$

So

$$x! = \sqrt{2\pi x} x^x e^{-x}\left\{1 + \frac{1}{12}x + \ldots\right\}$$

Can these things be crosschecked with anything?

The term $x \ln x - x$ that for integers could be $n \ln n - n$ is what we would get by approximating $\ln n!$, i.e. $\ln 1 + \ln 2 + \ldots + \ln n$ by $\int_1^n \ln x\, dx$. But there is more that can be crosschecked. For large integer n Stirling had obtained a similar series by using the Euler-McLaurin formula:

$$\ln n! = \ln 1 + \ln 2 + \ldots + \ln n$$
$$= \int_1^n \ln x\, dx + \frac{1}{2}\{\ln 1 + \ln 2\} + \frac{1}{12}\{(\ln x)'_{x=n} - (\ln x)'_{x=1}\}$$
$$- \frac{1}{720}\{(\ln x)'''_{x=n} - (\ln x)'''_{x=1}\}$$
$$= n \ln n - n + 1 + \frac{\ln n}{2} + \frac{1}{12}\left(\frac{1}{n} - 1\right) - \frac{1}{720}\left(\frac{2}{n^3} - 2\right) + \ldots \qquad (1.35)$$

Apart from a set of constants (and apart from $\frac{-2}{720n^3}$ of an order the previous result did not reach) the first few terms check (?????????????)

Can one go from $\ln n! = \text{const} + n \ln n - n + \frac{\ln n}{2}$ i.e. from $n! = e^{\text{const}}\, n^{n+\frac{1}{2}}\, e^{-n}$ to the value of this constant by using Wallis' formula of page 8, or by using the "duplication formula" we saw a couple of pages ago

$$\sqrt{\pi} = \lim_{n \to \infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{2n}}$$

Does it agree with the value $\sqrt{2\pi}$ of the previous page?

Let's also use the above to analyze[13] $\frac{N!}{m!(N-m)!} P^m (1-P)^{N-m}$ when all m, N, $N-m$ are large.

....

Postponing for somewhat longer (for quite a while actually) the treatment of the issue of how to justify rigorously the addition order by order of an infinite number of power series (that was one of the needed steps in turning infinite partial fraction expansions into power series in pages 8,9) let's now (p.26) start facing the problem of the rigorous foundation of limits, continuity etc that we postponed in page 7. If, right after experience with these issues, the reader wants to see full details for the rigorous treatment of the plausible steps whose proof we skipped when we were doing the proof that the two definitions of the gamma function we saw referred to the same object, he/she can go to Spain and Smith's "Functions of mathematical physics" (Van Nostrand 1970) or google with *Bohr-Mollerup.*

[13] encountered in probability problems, like what is the probability of m heads in N tosses when each head has probability P. Or the probability m out of N molecules are in a subvolume v of a box of a volume V (if the conditions are uniform then $P = \frac{v}{V}$). We expect $m = N\frac{v}{V} = NP$. How sharp is the probability maximum at this value?The above can be estimated by Stirling's formula if N, m, $N-m$ are large.

Appendix A

Exercising in Rigor

Let's go to another issue that we have been postponing (while waiting for some more samples of results that mathematical physics was arriving at): the issue of more rigorous proofs of such results that up to this page, when suspect, were only checked through crosscheckings with alternative proofs.Let's start with an amusing informal exercise, hardly needing a pen: After reading a little about things like "foundations of real number systems", "greatest lower bound axiom", "greatest lower bound theorem", "Dedekind cuts", "nested intervals", "Bolzano-Weierstrass theorem for sequences",[1] "Bolzano-Weierstrass theorem for continuous functions", "fundamental theorem of algebra"[2] (about all polynomials having at least one root on the complex plane), say whether the following proofs are correct; if they are incorrect point out where; and if they can become correct by adding some steps add them:

a A continuous function changing sign in an interval must have a zero there because if it is nowhere zero then, divided by its absolute value, which is also continuous, the ratio, which too is continuous, would only have as values 1 and -1, which with such a wide gap cannot be values of a continuous function.

b If by handedness of a system of three orthogonal 3-dim vectors we mean the sign of the 3×3 determinant of their components (which, in absolute value, would be the volume ofthe rectangular parallelepiped the three vectors form) then a continuous transformation preserving lengths and angles coud

[1]We especially recommend K. Knopp's small book "Infinite Sequences and Series" (Dover 1956) and also the first chapter of his bigger book "Theory and Applications of Infinite Series" (Dover 1990) to other chapters of which we have already referred the reader.

[2]We especially recommend Courant and Robbins' "What is Mathematics" (recently revised by Ian Stewart).

not start from the original triad and end in one of opposite handedness because it would have to make the determinant pass from zero whereas the volume of three orthogonal vectors of fixed length cannot be zero.

c Another proof (or "proof"?) of the "Bolzano-Weierstrass theorem for continuous functions" that we saw in part a. above, could go as follows: Suppose a continuous function changes sign in the interval $(0, 1)$. Divide the interval in ten equal sub- intervals. In at least one of them the function changes sign. Take that sub-interval and consider its leftest point as the tenth's digit of a number. That sub-interval's numbers start with that tenth's digit. Divide that subinterval in ten equal sub-sub-intervals. In at least one of them too the function would change sign. Take its leftest point and consider it as the hundreds' digit of a number etc etc. The number constructed in that way is a number at which the continuous function is zero because in the vicinity of that number exist both a positive and a negative number, no matter if that vicinity has length of one tenth, one hundredth etc etc. If the value of the function at that point was positive then we would be able to find a small enough vicinity of it with only positive values, and the same for the negative case. So the value of the function there is zero.

d An increasing sequence that is bounded from above is convergent because: Take a bound and suppose it starts from some digit of say, tens (it could be millions but say tens). Divide the real line (y-axis) in intervals of tens starting from zero and going below and above. Since the sequence is bounded from above we will not be seeing terms of it in all the above intervals. Take the last interval above zero where one still sees terms of the sequence, say it's the sixth, then write 5 as tens' digit. Now divide this sixth interval in ten equal sub-intervals, take the last one above where one still sees terms of the sequence, say it's the 8^{th}, write 7 as units' digit. Then divide...etc. The number we have written is a number to which the bounded sequence we had started with, converges because being increasing it cannot get back away from the intervals reached by terms of it, and also it approaches it to distances less than tenths, hundredths...etc etc.

e Concerning the proof of a well known theorem called "Cauchy's criterion" that states that if a sequence is such that for every positive number ϵ there is a positive integer N such that for m, n bigger than N the distance between the m^{th} term and the n^{th} term of the sequence is smaller than ϵ, then the sequence converges: Take N big enough to make that distance less than one tenth. After that order all the terms of the sequence will have the same digits up to the tenths'. Then take another integer N' big enough to make all the terms of the sequence of order greater than it have distance between them smaller than one hundredth...etc etc. The number we are constructing is anumber to which all the terms of the sequence gather, to distance smaller than one tenth , one hundredth etc etc, after an order high enough, so the sequence converges to that number.

f A proof of the Bolzano-Weierstrass theorems and of Cauchy's criterion that does not make use of the least upper-bound axiom either tacitly assumes it or is wrong because otherwise it could be used to prove that axiom but then we would not call it an axiom but a theorem.

g Suppose there existed a polynomial $p(z)$ that nowhere on the complex plane became equal to zero. Take a circle of radius R around the origin and make the complex number z traverse that circle once. The complex number $p(z)$ traverses a closed curve too and since it never becomes zero, thus never passes through the origin, it makes sense to say that it made a given number of windings around it. For small values of R the polynomial is like its constant term and thus the winding number is zero ($p(z)$ makes no rotation around the origin). For big values of R the polynomial is like its highest term z^n and thus, like $z^n = e^{in\theta}$, it makes n windings. The winding number should not change much if we don't change R much, thus it should be a continuous function of R, but being an integer it can't jump from one integer to another without violating the Bolzano-Weierstrass theorem that also states that a continuous function can't take two values without taking all the intermediate values somewhere in between. But then how did it jump from zero to n as we were increasing R? So we must have made a wrong assumption, so we wrongly assumed $p(z)$ never becomes zero. So all polynomials have at least one complex root.

Concerning the question of when one can add term by term an infinite number of infinite series, that we mentioned on page 25, one can go to even the *small* book by Knopp mentioned in the 1^{st} footnote of page 26 (the inclusion of this issue plus the inclusion of the *proof* of Gauss' very strong test for convergence of series (through comparison with harmonic series rather than geometric series like Cauchy's ratio test and Cauchy's root test) still makes this book unique among other elementary books). Another approach to the above term by term summation can be looked up in Caratheodory's "Theory of Functions"...

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